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Dominant of Functions Satisfying a Differential Subordination and Applications

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Abstract. Best dominant is obtained for normalized analytic functions *f* satisfying $(1-\alpha) f(z)/z+\alpha f'(z)+\beta z f''(z) \prec h(z)$ in the unit disk $\mathbb{D}$, where *h* is a normalized convex function, and α, β are appropriate real parameters. This fundamental result is next applied to investigate the convexity and starlikeness of the image domains $f(\mathbb{D})$ for particular choices of *h*.

Keywords: Starlike and convex functions, differential subordination, dominant.
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INTRODUCTION

Let $\mathcal{H}$ be the class of analytic functions f defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, n a positive integer, and $z \in \mathbb{D}$, let

$$\mathcal{H}_n(a) = \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\},$$

with $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of starlike functions in $\mathbb{D}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D},$$

is denoted by $\mathcal{ST}$, and $\mathcal{CV}$ is the subclass of $\mathcal{A}$ consisting of convex functions in $\mathbb{D}$ satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}.$$

For two analytic functions f and g, the function f is subordinate to g, written $f(z) \prec g(z)$ if there is an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. If g is univalent, then f subordinate to g is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

This paper considers a class of functions satisfying a second-order differential subordination to a given convex function. Best dominant amongst the solutions to this differential subordination is determined. Further, sufficient conditions are obtained that ensure these solutions are either starlike or convex functions in $\mathbb{D}$. Such conditions in terms of differential inequalities have been investigated in several works, notably by [1, 2, 3, 4, 5, 6, 7]. In particular, Kanas and Owa [8] studied connections between certain second-order differential subordination involving expressions of the form $f(z)/z$, $f'(z)$ and $1+zf''(z)/f'(z)$. The class studied in this paper presents a more general framework.

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The following lemma will be needed.

Lemma 1 [9, Theorem 1, p. 192] *Let* h *be convex in* $\mathbb{D}$ *with* $h(0)=a, \gamma \neq 0$ *and* $\operatorname{Re}\gamma \geq 0$. *If* $p \in \mathcal{H}_n(a)$ *and*

$$p(z)+\frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{(\gamma/n)-1}dt.$$

The function q *is convex and is the best* $(a,n)-$*dominant.*

SECOND – ORDER DIFFERENTIAL SUBORDINATION

In the following sequel, we shall assume that h is an analytic convex function in $\mathbb{D}$ with $h(0)=1$. For $\beta \geq 0$ and $\alpha+2\beta \geq 0$ consider the class of functions $f \in \mathcal{A}_n$ satisfying the second-order differential subordination

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta zf''(z) \prec h(z). \tag{1}$$

Let μ and ν satisfy

$$\nu+\mu=\alpha+\beta \quad \text{and} \quad \mu\nu=\beta. \tag{2}$$

Note that $\operatorname{Re}\mu \geq 0$ and $\operatorname{Re}\nu \geq 0$.

The following result gives the best dominant to solutions of the differential subordination (1).

Theorem 1 *Let* μ *and* ν *be given by (2), and* α, β *be real numbers such that* $\beta \geq 0$ *and* $\alpha+2\beta \geq 0$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta zf''(z) \prec h(z),$$

then

$$\frac{f(z)}{z} \prec q(z) := \frac{1}{\beta n^2}\int_0^1\int_0^1 h(rsz)r^{(1/\mu n)-1}s^{(1/\nu n)-1}drds,$$

and q *is the best* $(a,n)-$*dominant.*

Proof. Let

$$p(z)=\frac{f(z)}{z}=1+a_{n+1}z^n+a_{n+2}z^{n+1}+\cdots.$$

Evidently

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta zf''(z)=\beta z^2p''(z)+(\alpha+2\beta)zp'(z)+p(z),$$

and (1) can be expressed as

$$\beta z^2p''(z)+(\alpha+2\beta)zp'(z)+p(z) \prec h(z). \tag{3}$$

Writing

$$F(z) = \nu zp'(z) + p(z),$$

it follows that

$$F(z) + \mu zF'(z) = \beta z^2 p''(z) + (\alpha + 2\beta)zp'(z) + p(z) \prec h(z),$$

where μ and ν are given by (2). Lemma 1 now yields

$$F(z) \prec \frac{1}{\mu n z^{1/\mu n}} \int_0^z h(t) t^{(1/\mu n)-1} dt,$$

and thus

$$p(z) + \nu zp'(z) = \frac{1}{\mu n} \int_0^1 h(rz) r^{(1/\mu n)-1} dr.$$

A second application of Lemma 1 shows that

$$p(z) \prec \frac{1}{\nu n z^{1/\nu n}} \int_0^z \left(\frac{1}{\mu n} \int_0^1 h(rt) r^{(1/\mu n)-1} dr \right) t^{(1/\nu n)-1} dt,$$

which in view of (2) implies that

$$\frac{f(z)}{z} \prec q(z) := \frac{1}{\beta n^2} \int_0^1 \int_0^1 h(rsz) r^{(1/\mu n)-1} s^{(1/\nu n)-1} drds.$$

Since $q(z) + (\alpha + 2\beta)nzq'(z) + \beta\left[n(n-1)zq'(z) + n^2 z^2 q''(z)\right] = h(z)$, the function $Q(z) = zq(z)$ is a solution of the differential subordination

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta zf''(z) \prec h(z).$$

This shows that $q \prec \tilde{q}$ for all $(a, n)-$dominants $\tilde{q}$, and hence q is the best $(a, n)-$dominant. □

The following result is an immediate consequence of Theorem 1.

Corollary 1 *Under the assumptions of Theorem 1, if*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta zf''(z) \prec 1 + \mathrm{M}z, \tag{4}$$

then

$$\frac{f(z)}{z} \prec 1 + \frac{Mz}{1 + \alpha n + \beta n(n+1)}, \tag{5}$$

and the superordinate function is the best dominant.

An application of Corollary 1 gives the following sufficient condition for starlikeness.

Theorem 2 *Let α and β be real numbers with $\alpha \geq 1$ and $\beta \geq 2\alpha$. Further let $f \in \mathcal{A}_n$ and $0 < M < M(\alpha, \beta, n)$, where*

$$M(\alpha, \beta, n) = \frac{2(\beta - 2\alpha)[1 + \alpha n + \beta n(n+1)]}{\alpha(n-1) + \beta[n(n+1) + 2]}. \tag{6}$$

If f satisfies the differential subordination

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) + \beta zf''(z) \prec 1 + \mathrm{M}z,$$

then $f \in \mathcal{ST}$.

Proof. Let

$$f(z) = zw'(z). \tag{7}$$

A brief computation shows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(1 + \frac{zw''(z)}{w'(z)}\right). \tag{8}$$

In view of the analytical condition for starlikeness, that is, $\operatorname{Re}\left(zf'(z)/f(z)\right) > 0$ in $\mathbb{D}$, it remains to show that

$$\left|\frac{zw''(z)}{w'(z)}\right| < 1. \tag{9}$$

Using (7), (4) can be rewritten as

$$w'(z) + (\alpha + 2\beta)zw''(z) + \beta z^2 w'''(z) \prec 1 + Mz. \tag{10}$$

Integrating (10), evidently

$$(1-\alpha)w(z) + \alpha zw'(z) + \beta z^2 w''(z) = z + Mz\int_0^1 \phi(sz)ds, \tag{11}$$

where ϕ is an analytic self-map of $\mathbb{D}$ with $\phi(0) = 0.$ It follows from (7) and (11) that

$$\left|\frac{zw''(z)}{w'(z)}\right| \le \frac{1}{\beta}\left|\frac{1}{w'(z)}\right|\left|1 + M\int_0^1 \phi(sz)ds\right| + \frac{(\alpha-1)}{\beta}\left|\frac{w(z)}{zw'(z)}\right| + \frac{\alpha}{\beta}.$$

For (9) to hold true, it is sufficient to prove

$$\frac{1}{\beta}\left|\frac{1}{w'(z)}\right|\left|1 + M\int_0^1 \phi(sz)ds\right| + \frac{(\alpha-1)}{\beta}\left|\frac{w(z)}{zw'(z)}\right| + \frac{\alpha}{\beta} < 1. \tag{12}$$

Now the subordination (5), implies

$$\left|\frac{1}{w'(z)}\right| < \frac{1 + \alpha n + \beta n(n+1)}{1 + \alpha n + \beta n(n+1) - M}, \tag{13}$$

where $M < 1 + \alpha n + \beta n(n+1),$ while

$$\left|\frac{w(z)}{z}\right| < 1 + \frac{M}{2[1 + \alpha n + \beta n(n+1)]}. \tag{14}$$

Since $|\phi(z)| < |z|,$ a brief computation shows that

$$\left|1 + M\int_0^1 \phi(sz)ds\right| < 1 + \frac{M}{2}. \tag{15}$$

Taking into account the inequalities (13), (14) and (15), the condition (12) is fulfilled whenever $M < M(\alpha, \beta, n)$ with $M(\alpha, \beta, n)$ given by (6). This completes the proof. □

The following theorem which gives sufficient conditions for convexity is also a consequence of Corollary 1.

Theorem 3 *Let α and β be real numbers with $\alpha \geq 1$ and $\beta \geq \left(2+\sqrt{2}\right)\alpha$. Further let $f \in \mathcal{A}_n$ and $0 < M < M(\alpha, \beta, n)$, where*

$$M(\alpha,\beta,n) = \frac{2[1+\alpha n+\beta n(n+1)][(\beta-2\alpha)^2-2\alpha^2]}{2\beta[2+\alpha(n-1)+\beta n(n+1)]+(\beta-\alpha)[\alpha(n-1)+\beta n(n+1)]}. \tag{16}$$

If f satisfies the differential subordination

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta z f''(z) \prec 1+\mathrm{M}z,$$

then $f \in \mathcal{CV}$.

Proof. In view of the fact

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq 1-\left|\frac{zf''(z)}{f'(z)}\right| > 0,$$

it is sufficient to prove the inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1. \tag{17}$$

Let

$$f'(z)\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right] = zf''(z). \tag{18}$$

Proceeding similarly as in the proof of Theorem 2, with ϕ as an analytic self-map of the unit disk, it follows from (18) and (4) that

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta f'(z)\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right] = 1+M\phi(z). \tag{19}$$

Subsequently,

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1}{\beta}\left|\frac{1}{f'(z)}\right|\left|1+M\phi(z)\right| + \frac{(\alpha-1)}{\beta}\left|\frac{f}{zf'(z)}\right| + \frac{\alpha}{\beta},$$

which leads to the condition

$$\frac{1}{\beta}\left|\frac{1}{f'(z)}\right|\left|1+M\phi(z)\right| + \left(\frac{\alpha-1}{\beta}\right)\left|\frac{f}{zf'(z)}\right| + \frac{\alpha}{\beta} < 1 \tag{20}$$

for (17) to hold true.

Applying (7) and (11), as well as the inequalities (13), (14) and (15), yield

$$\left|\frac{1}{f'(z)}\right| < \frac{2\beta[1+\alpha n+\beta n(n+1)]}{2[1+\alpha n+\beta n(n+1)](\beta-2\alpha)-M[\alpha(n-1)+\beta(n(n+1)+2)]} \tag{21}$$

and

$$\left|\frac{f(z)}{zf'(z)}\right| < \frac{2\beta[1+\alpha n+\beta n(n+1)-M]}{2[1+\alpha n+\beta n(n+1)](\beta-2\alpha)-M[\alpha(n-1)+\beta(n(n+1)+2)]}, \tag{22}$$

where

$$M < \frac{2(\beta-2\alpha)[1+\alpha n+\beta n(n+1)]}{\alpha(n-1)+\beta[n(n+1)+2]}.$$

In view of (21), (22) and the fact that $|1+M\phi(z)|<1+M$, (20) is fulfilled for $M<M(\alpha,\beta,n)$, where $M(\alpha,\beta,n)$ is given by (16). This completes the proof. □

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